Computations of Powerball Odds

In this sidebar I want to explain how I computed the numbers in the following sentence:

> The chance of no winner is about 8%; of one winner, 21%; of two, 26%; of three, 21%; of four, 13%; of five or more, 9%.

Suppose the probability of winning the lottery is p. (For us, $p = 1/80,000,000$.) And suppose that the number of tickets purchased is N. (For us, $N = 200,000,000$.) What is the chance that exactly k winning tickets will be purchase?

To illustrate the method, let us ask the question for $k = 3$. Suppose we want to know what the chances are that my sister, my father, and my uncle, who bought one ticket each, will be the only three winners. The chance that my sister will win is p. The chance my father will win is p. Thechance my uncle will win is p. So there is a chance of p^3 that the three of them will all win. (Note that we've used here the **Product Rule** that I discussed in my last column, "Algebra for Adulterers.")

That's not quite enough; I also need it to be the case that the other $N - 3$ tickets do *not* win. Each of these tickets has a chance of $1 - p$ to be a loser. So the chance that all of them will lose is $(1 - p)^{N-3}$.

Putting these together, we find that the probability of my three family members holding the only winning tickets is

$$p^3(1 - p)^{N-3}$$

But when I compute the chance that there will be exactly three winners, I have to allow for *all* possible sets of three tickets. We now ask: How many sets of three tickets can be chosen from the N which were sold? There is an exact formula for this, but we'll use a harmless approximation and say there are approximately $(1/6)N^3$ such sets of three. So the probability that *one* of these sets of three tickets is the set of winners is

$$(1/6)N^3 \cdot p^3 \cdot (1 - p)^{N-3}.$$

When we plug in the values of p and N coming from Powerball, we get

$$(1/6)(200,000,000)^3 \cdot (1/80,000,000)^3 \cdot (79,999,999/80,000,000)^{199,999,997}$$

which your calculator will tell you is about .213, or 21%.

I still have to explain how I got $(1/6)N^3$ as the number of ways to pick 3 winners from N tickets. I'll justify it like this: the first winner can be any one of the N tickets. For each of these choices, the second winner can be any one of the remaining $N - 1$ tickets. So there are $N(N - 1)$ ways of choosing winners 1 and 2. Now for each of these $N(N - 1)$ choices, there are $N - 2$ ways to choose the third winner. So we get $N(N - 1)(N - 2)$ ways in all to choose the first three winners. But note that we counted winner 1 = my sister, winner 2 = my father, winner 3 = my uncle

and

winner 1 = my father, winner 2 = my uncle, winner 3 = my sister

separately, when in fact they are the same set of three people. In fact, we've counted this set of people 6 times, because there are 6 different orders in which three people can be put. So if we get an answer of $N(N-1)(N-2)$ when we count each possible set of three people 6 times, the number of sets of three people must be $(1/6)N(N-1)(N-2)$. Since N is pretty big, we can think of $N-1$ and $N-2$ as being "pretty close to N", and approximate the number of choices by $(1/6)N^3$.

The same reasoning suggests that the number of ways to choose k winners out of N tickets is approximately $N^k/k!$, where tht exclamation point is pronounced "factorial". The number $k!$ is just the number of orders into which k different objects can be put. So $2! = 2$, since two objects can only be put in two different orders. And $3! = 6$, as we've already discussed. You might enjoy trying to prove the formula

$$k! = k \cdot k - 1 \cdot k - 2 \cdot \ldots \cdot 2 \cdot 1.$$

We now find that, in general, the chance of there being exactly k winners is about

$$(1/k!) \cdot N^k \cdot p^k \cdot (1-p)^{N-k}.$$

Plugging in $p = 1/80,000,000$ and $N = 200,000,000$ yields the numbers given in the article.

One more thing: I said in the article that your expected share of the jackpot, if you win Powerball, is 37%. To get this, I had to use a little calculus.

Let's suppose you win. Hurray! Now the big question is, how many people *besides* you are going to win? This is the same problem as the one above, except there are now $N-1$ tickets–every one except yours–we have to worry about.

For simplicity, write x to denote the product Np.

Now the chance that there will be exactly k winners is approximately

$$(1/k!)x^k \cdot (1-p)^{(N-1)-k}.$$

The rule of thumb in the article tells us that $(1-p)^{(N-1)-k}$ is approximately e^{-Np}, or e^{-x}. So the chance that there will be exactly k winners besides you is about

$$e^{-x}(1/k!)x^k.$$

In that case, you get a share of $1/(k+1)$ of the jackpot.

When we discussed expected value in the article, we said that an object with a p_1 chance of having value v_1 and a p_2 chance of having value v_2 had an expected value of $p_1v_1 + p_2v_2$. Now the value of our share of the jackpot has, not, two, but *infinitely* many possible values. (Well, technically, the number of possibilities is bounded by the total number of people playing–but 200,000,000 is more or less infinite, right?) Fortunately, our formula for expected value generalizes in the natural way. Let $p(k)$ be the probability of k other winners; that is, the probability that your share is $1/(k+1)$ of the jackpot. Then the expected value of your share of the jackpot, should you win, is

$$p(0) \cdot 1 + p(1) \cdot (1/2) + p(2) \cdot (1/3) + \ldots$$

which, by the calculation above, is

$$e^{-x}(1/0!)x^0 \cdot 1 + e^{-x}(x/1!) \cdot (1/2) + e^{-x}(x^2/2!) \cdot (1/3) \ldots = e^{-x}x^0/1! + e^{-x}x^1/2! + e^{-x}x^2/3! \ldots$$

which all my calculus students will recognize as being equal, by Taylor's theorem, to

$$(1/x)(1 - e^{-x}).$$

Remember that in our case, $x = Np = 200,000,000/80,000,000 = 2.5$. So the expected value of your share of the jackpot is

$$(1/2.5)(1 - e^{-2.5}) = 36.7\%$$

as claimed in the article.